Stirling Numbers of the Second Kind

An Intro to Combinatorics

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The Ice Cream Problem

Question

Suppose we have k flavors of ice cream and n kids. How many ways can we serve the n kids using **all** k flavors?

We will answer this question, and more, using

Stirling Numbers of the Second Kind!

Outline

Sets

Definition

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Examples

1. $\{a, b, c, d\}$,
2. the set of even numbers $\{0, 2, -2, 4, -4, 6, -6, \dots\}$, and
3. the set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \dots\}$.

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Definition

Given A is a set, we let $|A|$ denote the number of elements of A, or the *cardinality* of A.

Combinations

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One can think about this via the following example:

Example

There are $\binom{7}{3} = \binom{7}{4}$ ways to rearrange the numbers in the word 0001111, where either you choose where the three zeros go, or where the four ones go.

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We can better understand this using an example:

Example

The set $\{a, b, c\}$ has 5 partitions:

1. $\{a\}, \{b\}, \{c\}$,
2. $\{a, b\}, \{c\}$,
3. $\{a, c\}, \{b\}$,
4. $\{a\}, \{b, c\}$, and
5. $\{a, b, c\}$.

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Example

Given the set $\{a, b, c, d\}$, if we wanted to find the number of ways to distribute the letters into 3 nonempty subsets, such as:

$\{a, b\}, \{c\}, \{d\}$

$\{a, c\}, \{b\}, \{d\}$

$\vdots$

we would obtain $\left\{ 4 \atop 3 \right\} = 6$

Stirling Number Identities

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Given a set with n elements, the only way to create one nonempty subset using all of the elements would be the set itself. $\square$

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Proof.

Considering a set with n elements that we want to distribute into $n-1$ nonempty subsets, using the Pigeonhole Principle, we know that 2 elements must be in the same subset. The number of ways to choose these 2 elements from our total n is $\binom{n}{2}$. $\square$

Pascal's Identity

Identity 3

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Proof by Example

Suppose we have 18 students, and we want to choose 11 for a committee. Jane, one of the 18 students, has two options, she can either be in the committee or not in the committee. If she were in the committee, we would have to pick 10 out of the other 17 to also be in the committee. Otherwise, we have to pick 11 out of the other 17. This can be written out as

$$\binom{17}{10} + \binom{17}{11} = \binom{18}{11}.$$

Pascal's Identity's for Stirling Numbers

Given Pascal's Identity, we wonder if a similar identity could exist for Stirling numbers of the second kind. From this, we come with the proposition:

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Proof

Suppose we have set with $n + 1$ elements, and we want to distribute them into k nonempty subsets. For a specific element, there are two cases: it's either alone in a subset or in a subset with other elements. If it's in its own subset, we have to distribute the other n elements into $k - 1$ subsets. Otherwise, we first distribute the other n elements into all k subsets, then multiply by k to choose which subset to put our original element in.

Stirling Numbers in the Ice Cream Problem

Problem

How many ways are there to give n kids k ice creams using all of the flavors?

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Solution

We distribute the n kids into k subsets, where each subset gets a different flavor. Then, we have to multiply this by $k!$ since the flavors distinct and ordered

$$k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

This is unsatisfying to us though, because we still don't know how to solve for $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.

Principle of Inclusion-Exclusion

Diagram

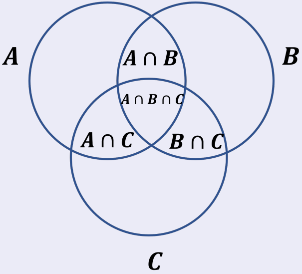

Figure: In order to find $|A \cup B \cup C|$, we can consider $|A| + |B| + |C|$, but then we've overcounted by adding $|A \cap B|, |B \cap C|$, and $|A \cap C|$. We have to subtract these, so we get $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C|$. Then, we've undercounted $|A \cap B \cap C|$ so we have to add it back. Finally, we obtain

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

Principle of Inclusion-Exclusion

The previous slide demonstrates and motivates the *Principle of Inclusion-Exclusion (PIE)*.

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Principle

PIE helps us determine the number of elements in the union of a group of sets. It states that: Given finite sets $A_1, \ldots, A_n$, one has the following identity:

$$|\cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{1\leq i<j\leq n} |A_i \cap A_j| + \sum_{1\leq i<j<j\leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

Where we add the cardinality of each of the sets and alternate subtracting and adding the intersections when we overcount and undercount.

PIE in the Ice Cream Problem

The total amount of ways to give n kids k flavors of ice cream is k^n. We multiply k by itself n times because for each of the n kids there are k possibilities.

This overcounts the cases where we don't use at least one of the flavors, since the problem asks for the cases were we use *all* of the flavors.

We first subtract the cases for each of the k flavors where we don't use that flavor:

$$k^n - \binom{k}{1}(k-1)^n$$

But then we've subtract the cases where we don't use two of the flavors twice, so we have to add those cases back:

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n$$

.

Solution to the Ice Cream Problem Without Stirling Numbers

Solution

This pattern continues, so in the end, we get that the number of ways to give n kids k flavors with all flavors used is:

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \ldots + (-1)^k \binom{k}{k}(k-k)^n$$

We can also write this as:

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r}(k-r)^n$$

.

Stirling Numbers of the Second Kind Formula

Now that we've solved the problem without Stirling numbers, we can set them equal to each other and isolate $\left\{ {n \atop k} \right\}$ to find a formula for Stirling numbers of the second kind.

$$k!\left\{ {n \atop k} \right\} = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n$$

. Dividing both sides by $k!$, we get:

Formula

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n$$

.

Stirling Numbers Formula Applied

Problem

Assume PRIMES Circle has 20 students and they are serving 5 different flavors of ice cream. How many ways are there to distribute the ice cream using all of the flavors?

Solution

We can just plug these numbers into our formula now!

$$5!\left\{ {20 \atop 5} \right\} = 5! \times \frac{1}{5!} \sum_{r=0}^{5} (-1)^r \binom{5}{r} (5-r)^{20}$$
$$= 89904730860000$$

WOAH!

Thank you for listening!

Any Questions?